MATEMATIKA FANINI KASBGA YO'NALTIRIB O'QITISHDA ANIQ INTEGRALNING BA'ZI TADBIQLARI

Samarqand viloyat Oqdaryo tuman 2-son kasb-hunar maktabi
matematika fani o'qituvchisi
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Annotatsiya: Ushbu maqolada aniq integralning ba'zi tadbiqlari jumladan yuzlarni integrallar yordamida hisoblash, aylanish jismlarining hajmini integrallar yordamida hisoblash ko'rib chiqiladi. Egri chiziqli trapetsiya yuzi, har xil figuralarning yuzlari, konus hajmi, kesik konus hajmi, sharning hajmi, shar segmentining hajmi, silindrning hajmini hisoblashga doir misollarning ishlanish usullari o'rganiladi va professional ta'lim o'quvchilariga o'rgatishning qulay metodikasi tahlil qilinadi.

Kalit so'zlar: Aniq integral, uzluksiz funksiya,egri chiziqli figura, ellipsning yuzi, 2 ta parabolakesishidan hosil bo'lgan figura yuzi, qutb koordinatalarida yuza hisoblash, paraboloid segmentning hajmi, aylanish jismning hajmi,sikloidaning yuzi, segmentning hajmi

Yassi figuralarning yuzini hisoblash

Yassi figuralarning yuzini hisoblashda aniq integralni qo`llashning bir necha hollari mavjud. Bunda chegara funksiyalarining joylashuv vaziyatlari muhim ahamiyatga ega. Ba`zi hollarini ko`rib o`tamiz.

1)Agar $y = f(x)$ funksiya OX o`qining yuqori (manfiy bo`lmagan) qismida joylashgan hamda uzluksiz bo`lib, $x = a$ va $x = b$ to`g`ri chiziq kesmalari bilan chegaralangan bo`lsa, hosil bo`lgan egri chiziqli trapetsiya yuzi

$$S = \int_a^b y dx \quad \text{yoki} \quad S = \int_a^b f(x) dx \qquad (1)$$

formula yordamida topiladi.

Misol: $y = x^2 + 1$, $y = 0$, $x = -1$, $x = 2$ chiziqlar bilan chegaralangan figuraning yuzini hisoblang.

Yechilishi: Shartga asosan figura $y = x^2 + 1$ egri chiziq, absissalar o`qi ( $y = 0$ ) hamda $x = -1$ **va** $x = 2$ to`g`ri chiziqlar bilan chegaralangan. U holda, (1) formuladan foydalanib, quyidagi integralni hisoblaymiz:

$$S = \int_{-1}^{2} (x^2 + 1) dx = \left(\frac{x^3}{3} + x \right) \Bigg|_{-1}^{2} = \frac{2^3}{3} - \frac{(-1)^3}{3} + 2 - (-1) = 6.$$

Demak, berilgan egri chiziqli trapesiyasimon figuraning yuzi 6 ga teng ekan.

2) Agar $y = f(x)$ funksiya OX o`qining pastki qismida joylashgan hamda uzluksiz bo`lib, $x = a$ va $x = b$ to`g`ri chiziq kesmalari bilan chegaralangan bo`lsa, hosil bo`lgan egri chiziqli trapetsiyasimon quyidagi formula yordamida topiladi:

$$S = -\int_{a}^{b} y dx \quad \text{yoki} \quad S = -\int_{a}^{b} f(x) dx. \qquad (1)$$

Misol: $y = -x^2 - 2$, $y = 0$, $x = -1$, $x = 1$ chiziqlar bilan chegaralangan figuraning yuzini hisoblang.

Yechilishi: Berilgan masalani yechish uchun (2) formuladan foydalanib, chegaralari -1 va 1 dan iborat bo`lgan quyidagi aniq integralni hisoblaymiz:

$$S = -\int_{-1}^{1} (-x^2 - 2) dx = \int_{-1}^{3} (x^2 + 2) dx = \left(\frac{x^3}{3} + 2x \right) \Bigg|_{-1}^{1} = 3\frac{2}{3}.$$

3) $y = f(x)$ uzluksiz funksiya grafigi $[a, b]$ kesmada OX o`qini chekli sondagi nuqtalarda kesib o`tsin. U holda, $[a, b]$ kesma funksiyaning ishorasi almashinishiga asoslanib, bir xil ishorali qismlari alohida –alohida kesmachalarga ajratiladi, ya`ni $[a; c]$, $[c; d]$, $[d; e]$ va $[e, b]$. U holda izlangan $S$ yuza hosil bo`lgan yuzachalarning

algebraik yig`indisidan iborat bo`ladi. Bunda qism funksiyalarning ishoralari e`tiborda bo`ladi. Izlanayotgan S yuza quyidagi integrallarning algebraik yiqindilari yordamida topiladi:

$$S=-\int_a^c f(x)dx+\int_c^d f(x)dx-\int_d^e f(x)dx+\int_e^b f(x)dx. \qquad (3)$$

Misol: $y=x+2$ va $y=x^2$ chiziqlar bilan chegaralangan figuraning yuzini toping.

Yechilishi: Integrallash chegaralarini ya`ni $a$ va $b$ ni berilgan chiziq tenglamalarini o`zaro tenglashtirib, topamiz:

$$x+2=x^2, \quad x^2-x-2=0.$$

Bundan, $x_1=-2,\ x_2=1$ yani $a=-2,\ b=1.$ U holda, (4) formulaga asosan:

$$S=\int_{-2}^{1}\left((x+2)-x^2\right)dx=\left(\frac{x^2}{2}+2x-\frac{x^3}{3}\right)\Bigg|_{-2}^{1}=\frac{1}{2}-\frac{1}{2}+2-(-4)-\frac{1}{3}-\left(-\frac{8}{3}\right)=8\frac{1}{3}.$$

Demak, izlanayotgan figuraning yuzasi $S=8\frac{1}{3}$ dan iborat ekan.

Quyida ba`zi egri chiziqli figuralarning yuzalarini topish formulalarni qaraymiz.

Ellipsning yuzi

Ma`lumki, ellipsning tenglamasi

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad (5)$$

dan iborat. Ellipsni 4 ta chorakka ajratib, uning bir bo`lagi, ya`ni $\frac{1}{4}S$ ni topish yetarlidir. (5) ga asosan

$$y=\frac{b}{a}\sqrt{a^2-x^2}\,. \qquad (6)$$

(1) formulaga asosan $\frac{1}{4}S=\int_a^b ydx=\frac{b}{a}\int_0^a\sqrt{a^2-x^2}\,dx.$

Quyidagi almashtirishlar olamiz: $x=a\sin t,\quad dx=a\sin tdt.$

U holda, integralning yangi chegaralarini aniqlaymiz: $0=a\sin t$ va $a=a\sin t$ lardan $\alpha=0$ va $\beta=\frac{\pi}{2}.$ Bulardan ,

$$\frac{1}{4}S=\frac{b}{a}\int_0^{\frac{\pi}{2}}\sqrt{a^2-a^2\sin^2 t}\cdot a\cos tdt=ab\int_0^{\frac{\pi}{2}}\cos^2 tdt=\frac{ab}{2}\int_0^{\frac{\pi}{2}}(1+\cos 2t)dt=$$

$$=\frac{ab}{2}\left(1+\frac{1}{2}\sin 2t\right)\Bigg|_0^{\frac{\pi}{2}}=\frac{ab}{2}\cdot\frac{\pi}{2}=\frac{\pi}{4}ab.$$

Demak, $\frac{1}{4}S=\frac{\pi}{4}ab.$ Bundan, $S=\pi ab.$ (7)

Ellips yuzini topishning umumiy fomulasi quyidagi ko`rinishda bo`ladi:

$$S=2\int_{-a}^{a}\frac{b}{a}\sqrt{a^2-x^2}\,dx=\pi ab. \qquad (8)$$

Ikkita parbolaning kesishmasidan hosil bo`lgan figuraning yuzi

$y^2=2px$ va $x^2=2py$ parabolalar berilan. S yuza $OKAN$ va $OLAM$ yuzalar M $x^2=2py$ ayirmasiga teng. Berilgan paralar $O(0;0)$ va $A(2p;2p)$ nuqtalarda kesishishadi.

Shuning uchun

$$S=\int_0^{2p}\sqrt{2px}dx-\int_0^{2p}\frac{x^2}{2p}dx=\frac{4}{3}p^2=\frac{(2p)^2}{3}.$$

Demak, izlangan $OKALO$ yuza $OMAN$ kvadratining uchdan bir qismidan iborat.

Sikloidaning yuzi

$x = a(t - \sin t)$ va $y = a(1 - \cos t)$ berilgan bo`lsa,

$$S_{OMANO} = \int_0^{2\pi} y dx = a^2 \int_0^{2\pi} (1 - \cos t)^2 dt = 3\pi a^2. \quad (10)$$

Demak, sikloidaning yuzi $S = 3\pi a^2$ iborat ekan .

Qutb koordinatalarida yuzani topish

AOB sektor AB yoy, OA va OB nurlar bilan chegaralangan bo`lsin. Bunday sektorning yuzi qutb koordinatalarida quyidagi formula yordamida topiladi:

$$S = \frac{1}{2} \int_{\varphi_1}^{\varphi_2} r^2 d\varphi. \quad (11)$$

Bunda r -qutb radiusi, φ -qutb radiusining OX o`q bilan tashkil qilgan burchagi, ya`ni qutb burchagi.

Aylanish jismini hajmi

$y = f(x)$ formula bilan berilgan AB egri chiziqning $[a, b]$ kesmada OX o`qi atrofida aylanishidan hosil bo`lgan jismning hajmini topish talab qilinsin. Aylanish jismini OX ga perpendikulyar tekislikdar bilan n ta bo`laklarga ajratamiz. Perpendikulyar tekisliklarning biri 0 nuqtadan $a$ masofada, ikkinchi tekislik $x$ masofada, keyingisi esa $x + h$ masofada bo`lsin. Bunda, h - orttirma bo`lib, $h = dx$ dir. U holda, jismning birinchi ikki tekislik bilan kesilgan qismining hajmi $v(x)$, undan keyingi qismining hajmi esa $v(x) + \Delta v(x)$dan iborat bo`ladi. Birinchi silindrsimon jismning balandligi $h = dx$, asos radiusi $y = f(x)$; ikinchisining balandligi ham $h = dx$, asos radiusi $y + \Delta y$. U holda, birinchi jism hajmi $\pi y^2 dx$, ikkinchisiniki esa

$\pi(y+\Delta y)^2 dx$ bo`ladi. Ikki silindr orasidagi $\Delta v$ orttirma hajm $2\pi hy \cdot \Delta y$ dan iborat bo`ladi. Ammo Δv hajm $\Delta y \to 0$ va $h \to 0$ da cheksiz kichik miqdor bo`lib, 0ga intiladi. Shuning uchun hajmning differensiali kichik silindrsimon jismning hajmi $\pi y^2 dx$ bo`ladi. Buni integrallaymiz:

$$v = \int_a^b \pi y^2 dx = \pi \int_a^b \left(f(x)\right)^2 dx. \qquad (1)$$

(1) tenglik ***aylanish jismining hajmini*** topish formulasidan iborat.

1-misol. Asos radiusi $MN = r$ va balandligi $ON = h$ bo`lgan aylanish paraboloidi segmentining hajmini toping.

Yechilishi: Ma`lumki, parabola tenglamasi $y^2 = 2px$ bo`lib, parabolaning ixtiyoriy $N(h; r)$ nuqtadan o`tishini e`tiborga olsak.

$$r^2 = 2ph. \qquad (2)$$

Parabola tenglamasi va (2) dan $y = \frac{r}{\sqrt{h}}\sqrt{x}.$ (3)

bo`ladi. Bundan, $y^2 = \frac{r^2}{h}x.$ (4)

U holda, (1) formulaga asosan ***paraboloid segmentining hajmi*** quyidagicha bo`ladi:

$$v = \pi \int_a^b \frac{r^2 x}{h} dx = \frac{1}{2}\pi r^2 h. \qquad (5)$$

2-misol. $y = x^2$ parabola, OX o`q va $x = 1$ to`g`ri chiziq bilan chegaralangan egri chiziqli trapesiyaning $OX$ o`qi atrofida aylanishidan hosil bo`lgan jismning hajmini toping.

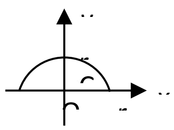
Yechilishi: (1) formuladan foydalanamiz. Bunda, $f(x)=x^2$, $a=0$ va $b=1$ larni formulaga qo`yib, integralni hisoblaymiz:

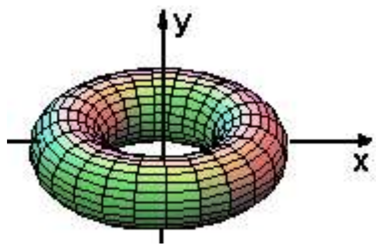
$$v=\pi\int_0^1 \left(f(x)\right)^2 dx = \pi\int_0^1 \left(x^2\right)^2 dx = \pi\int_0^1 x^4 dx = \pi\cdot\left.\frac{x^5}{5}\right|_0^1 = \frac{\pi}{5}.$$

Demak, jismning hajmi $\frac{\pi}{5}$ dan iborat ekan.

Guldining birinchi teoremasi). *Birjinsli tekis egri chiziqni u yotgan tekislikda olingan va uni kesmaydigan, biror o`q atrofida aylantirish natijasida hosil qilingan aylanish sirtining yuzi shu egri chiziq yoyning uzunligi bilan uning og`irlik markazi aylanishidan hosil bo`lgan aylana uzunligining ko`paytmasiga teng.*

misol. Radiusi *r* ga teng bo`lgan aylanani, uning tekisligida yotgan va uni kesmaydigan hamda aylana markazidan *d* masofada yotgan o`q atrofida aylantirish natijasida hosil qilingan *tor* sirtining yuzi va hajmi topilsin





Yechish. Masala shartiga ko`ra $d \geq r$. Aylananing uzunligi $2\pi r$ va u birjinsli ekanligidan uning og`irlik markazi o`zining markazidan iborat bo`lib, og`irlik markazini O*y* o`qi atrofida aylanishidan uzunligi $2\pi d$ ga teng bo`lgan aylana hosil bo`ladi. Demak, Guldinning birinchi teoremasiga ko`ra *tor* sirtining yuzi Q:

$$Q=(2\pi r)\cdot(2\pi d)=4\pi^2 rd$$

bo`ladi. Berilgan doiraning yuzi πr^2 ga teng. Uholda *tor* xajmi

$$V=\pi r^2\cdot(2\pi d)=4\pi^2 r^2 d$$

ga teng bo`ladi.

4. Ko`ndalang kesim yuzi ma`lum bo`lgan jismning hajmi

$x=a$ va $x=b$ kesmalardan o`tgan hamda $OX$ o`qqa perpendikulyar bo`lgan tekisliklar bilan chegaralangan $V$ jismning hajmini topish talab qilinsin. U holda jismni $n$ ta o`zaro parallel bo`lgan tekisliklar bilan $OX$ o`qiga parallel holda bo`laklarga ajratamiz. Ixtiyoriy $x=x_{i-1}$ va $x=x_i$ tekisliklar bilan chegaralangan jismning hajmi $\Delta v_i$, asos yuzi $S(x_i)$, balandligi $\Delta x_i$ bo`lsin.

U holda, $\Delta v_i = S(x_i)\cdot \Delta x_i$ (6)

o`rinli bo`ladi. Jismning umumiy hajmi quyidagicha bo`ladi:

$$v=\lim_{\varepsilon\to 0}\sum_{i=1}^{n} S(x)\Delta x_i \; . \qquad (7)$$

Bunda, ε balandlik Δx_i larning eng kattasi. (7) tenglik integral yig`indidan iborat. Shuning uchun (7) ni quyidagicha ifodalash mumkin:

$$v=\int_a^b S(x)dx. \qquad (8)$$

(8) - ***ko`ndalang kesim yuzi ma`lum bo`lgan jismning hajmini*** topish formulasi.

Misol. $y^2=4x,\ \ y=0,\ \ x=0,\ \ x=4$ chiziqlar bilan chegaralangan hamda OX o`q atrofida aylanishdan hosil bo`lgan jismning hajmini toping.

Yechilishi: Hosil bo`ladigan jism aylanish paraboloididan iborat bo`ladi. Uning hajmini (8) formula yordamida topamiz. Bunda $a=0$, $b=4$ va $S(x)=4x$ dir.

$$v = \pi \int_0^4 4x dx = \pi \cdot 4 \cdot \frac{x^2}{2}\Big|_0^4 = 2\pi x^2\Big|_0^4 = 32\pi.$$

Ba`zi jismlarning hajmini topish formulalari:

Piramidaning hajmi:

$$v = \frac{S}{H^2}\int_0^H x^2 dx = \frac{1}{3}SH.$$

Paraboloid segmentiining hajmi:

$$v = \pi \int_0^h \frac{r^2 x}{h} dx = \frac{1}{2}\pi r^2 h$$

(ya`ni silindr hajmining yarimiga teng- Arximed tadqiqoti)

Elliptik asosli konus:

$$v = \frac{1}{3}Sa \qquad (a \text{ - katta yarim o`q})$$

Ellipsoid:

$$v = \pi \int_a^b \frac{b^2}{a^2}\left(a^2 - x^2\right)dx = \frac{4}{3}\pi a b^3.$$

Sharning hajmi:

$$v = \pi \int_{-r}^{r} y^2 dx = \pi \int_{-r}^{r}\left(r^2 - x^2\right)dx = \frac{4}{3}\pi r^3.$$

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